Recapitulating the Basel Problem in Year 12: a history-as-a-tool approach to Euler's proof

Davide Leonessi

Leone Burton New Researchers' Day
British Society for Research into Learning Mathematics

King's College London, 13 June 2026

Plan

- Intro to History of Maths in teaching
- Lesson
  - Content and context
  - My approach: plan for worksheet and slides
  - Resources and delivery
    - ★ Slides
    - ★ Worksheet questions, with student work
- Evaluation of lesson
  - Students' feedback
  - Reflections
  - Possible extensions

Introduction: History of Maths in Teaching

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of “the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them.” (Blåsjö, 2014)

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of "the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them." (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of “the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them.” (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
-

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of "the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them." (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of "the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them." (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Recapitulation argument:

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of "the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them." (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Recapitulation argument: a learner repeats the development of Maths

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of "the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them." (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Recapitulation argument: a learner repeats the development of Maths

"Children should repeat the learning process of mankind, (sic) not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now" (Freudenthal, 2002)

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of “the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them.” (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Recapitulation argument: a learner repeats the development of Maths

“Children should repeat the learning process of mankind, (sic) not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now” (Freudenthal, 2002)

- early 1900s: Poincaré, Klein, and Severi
-

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of "the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them." (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Recapitulation argument: a learner repeats the development of Maths

"Children should repeat the learning process of mankind, (sic) not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now" (Freudenthal, 2002)

- early 1900s: Poincaré, Klein, and Severi
- experimental support, Bagni (2000,2005)

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of "the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them." (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Recapitulation argument: a learner repeats the development of Maths

"Children should repeat the learning process of mankind, (sic) not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now" (Freudenthal, 2002)

- early 1900s: Poincaré, Klein, and Severi
- experimental support, Bagni (2000,2005)

Mathematical Transgressions: cognitive boundaries and transgressive actions (Semadeni, 2015; Watford & Clark, 2024)

Introduction: History of Maths in Teaching

History of Maths in distinct traditions: Mathematics, Education, Philosophy, and History.

History of Maths is the study of “the development of mathematical ideas: (...) the interplay between them, and the ways in which they were understood and applied by the people who explored them.” (Blåsjö, 2014)

Teaching approaches: (Jankvist, 2009)

- History as a cognitive and affective *tool*
- History as a *goal*

Recapitulation argument: a learner repeats the development of Maths

“Children should repeat the learning process of mankind, (sic) not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now” (Freudenthal, 2002)

- early 1900s: Poincaré, Klein, and Severi
- experimental support, Bagni (2000,2005)

Mathematical Transgressions: cognitive boundaries and transgressive actions (Semadeni, 2015; Watford & Clark, 2024)

Criticisms: abundant, including the lack of empirical evidence and teachers’ preparation

Content, Context, and Approach

Lesson topic: the Basel Problem, i.e. computing $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Content, Context, and Approach

Lesson topic: the Basel Problem, i.e. computing $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Context:

- Year 12 Further Maths class
- King's Maths School, a selective state-funded Sixth-form

Content, Context, and Approach

Lesson topic: the Basel Problem, i.e. computing $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Context:

- Year 12 Further Maths class
- King's Maths School, a selective state-funded Sixth-form
- Students familiar with:
  - Finite and infinite geometric series
  -

Content, Context, and Approach

Lesson topic: the Basel Problem, i.e. computing $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Context:

- Year 12 Further Maths class
- King's Maths School, a selective state-funded Sixth-form
- Students familiar with:
  - Finite and infinite geometric series
  - Worksheet-based teaching

My teaching approach

Lesson preparation:

- Review of historical literature: focus on accessible maths

My teaching approach

Lesson preparation:

- Review of historical literature: focus on accessible maths
- Selection of questions, chronologically ordered

My teaching approach

Lesson preparation:

- Review of historical literature: focus on accessible maths
- Selection of questions, chronologically ordered
- Slides for historical context, *without answer to the title problem*

My teaching approach

Lesson preparation:

- Review of historical literature: focus on accessible maths
- Selection of questions, chronologically ordered
- Slides for historical context, *without answer to the title problem*
- Asking fellow trainees to work through draft: Linjia Chen's feedback particularly helpful

My teaching approach

Lesson preparation:

- Review of historical literature: focus on accessible maths
- Selection of questions, chronologically ordered
- Slides for historical context, *without answer to the title problem*
- Asking fellow trainees to work through draft: Linjia Chen's feedback particularly helpful

Lesson delivery:

- 10 minutes lecturing with slides

My teaching approach

Lesson preparation:

- Review of historical literature: focus on accessible maths
- Selection of questions, chronologically ordered
- Slides for historical context, *without answer to the title problem*
- Asking fellow trainees to work through draft: Linjia Chen's feedback particularly helpful

Lesson delivery:

- 10 minutes lecturing with slides
- 50 minutes for independent/pair work on worksheet: pacing given by frequent modelling

The Basel Problem:

The quest to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Mr Leonessi

Inspired by a conversation with Jason Yip
at the Annual Meeting of the British Society for the History of Mathematics

King's College London Maths School, March 2026

The Basel Problem: a timeline

The Basel Problem: a timeline

$\sim$1360: Nicole Oresme (1325–1382), in Paris, proves that $\sum_{n\geq 1}\frac{1}{n} = \infty$.

The Basel Problem: a timeline

$\sim$1360: Nicole Oresme (1325–1382), in Paris, proves that $\sum_{n\geq 1}\frac{1}{n}=\infty$.

1650: Pietro Mengoli (1626–1686), in Bologna, poses the Basel Problem: what is the value of $\sum_{n\geq 1}\frac{1}{n^2}$?

The Basel Problem: a timeline

$\sim$1360: Nicole Oresme (1325–1382), in Paris, proves that $\sum_{n\geq 1} \frac{1}{n} = \infty$.

1650: Pietro Mengoli (1626–1686), in Bologna, poses the Basel Problem: what is the value of $\sum_{n\geq 1} \frac{1}{n^2}$?

1689: Johann Bernoulli (1667–1748), in Basel, shows that $\sum_{n\geq 1} \frac{1}{n^2+n} = 1$.

The Basel Problem: a timeline

$\sim$1360: Nicole Oresme (1325–1382), in Paris, proves that $\sum_{n\geq 1} \frac{1}{n} = \infty$.

1650: Pietro Mengoli (1626–1686), in Bologna, poses the Basel Problem: what is the value of $\sum_{n\geq 1} \frac{1}{n^2}$?

1689: Johann Bernoulli (1667–1748), in Basel, shows that $\sum_{n\geq 1} \frac{1}{n^2+n} = 1$.

He also shows that $\sum_{n\geq 1} \frac{1}{n^2} \leq 2$ and observes that the Basel Problem is "more difficult than one would expect."

The Basel Problem: a timeline

$\sim$1360: Nicole Oresme (1325–1382), in Paris, proves that $\sum_{n\geq 1} \frac{1}{n} = \infty$.

1650: Pietro Mengoli (1626–1686), in Bologna, poses the Basel Problem: what is the value of $\sum_{n\geq 1} \frac{1}{n^2}$?

1689: Johann Bernoulli (1667–1748), in Basel, shows that $\sum_{n\geq 1} \frac{1}{n^2+n} = 1$.

He also shows that $\sum_{n\geq 1} \frac{1}{n^2} \leq 2$ and observes that the Basel Problem is "more difficult than one would expect."

1728: Christian Goldbach (1690–1764), in St Petersburg, *probably* proves that $\sum_{n\geq 1} \frac{1}{n^2} \leq 5/3$.

The Basel Problem: a timeline

$\sim$1360: Nicole Oresme (1325–1382), in Paris, proves that $\sum_{n\geq 1}\frac{1}{n}=\infty$.

1650: Pietro Mengoli (1626–1686), in Bologna, poses the Basel Problem: what is the value of $\sum_{n\geq 1}\frac{1}{n^2}$?

1689: Johann Bernoulli (1667–1748), in Basel, shows that $\sum_{n\geq 1}\frac{1}{n^2+n}=1$.

He also shows that $\sum_{n\geq 1}\frac{1}{n^2}\leq 2$ and observes that the Basel Problem is "more difficult than one would expect."

1728: Christian Goldbach (1690–1764), in St Petersburg, *probably* proves that $\sum_{n\geq 1}\frac{1}{n^2}\leq 5/3$.

In the meanwhile, the Basel Problem is discussed by:

John Wallis (1616–1703)
Gottfried Wilhelm Leibniz (1646–1716)
Jacob Bernoulli (1654–1705)
James Stirling (1692–1770)

The Basel Problem: a timeline (continued)

1650: P. Mengoli poses the Basel Problem: what is the value of $\sum_{n\geq 1}\frac{1}{n^2}$?

Leonhard Euler (1707–1783), in St Petersburg

The Basel Problem: a timeline (continued)

1650: P. Mengoli poses the Basel Problem: what is the value of $\sum_{n\geq 1} \frac{1}{n^2}$?

Leonhard Euler (1707–1783), in St Petersburg

1731: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to 3 decimal digits.

The Basel Problem: a timeline (continued)

1650: P. Mengoli poses the Basel Problem: what is the value of $\sum_{n\geq 1}\frac{1}{n^2}$?

Leonhard Euler (1707–1783), in St Petersburg

1731: computes $\sum_{n\geq 1}\frac{1}{n^2}$ to 3 decimal digits.

1733: computes $\sum_{n\geq 1}\frac{1}{n^2}$ to 20 decimal digits.

The Basel Problem: a timeline (continued)

1650: P. Mengoli poses the Basel Problem: what is the value of $\sum_{n\geq 1} \frac{1}{n^2}$?

Leonhard Euler (1707–1783), in St Petersburg

1731: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to 3 decimal digits.

1733: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to 20 decimal digits.

1734: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to “infinitely many” decimal digits. First proof of the Basel Problem.

The Basel Problem: a timeline (continued)

1650: P. Mengoli poses the Basel Problem: what is the value of $\sum_{n\geq 1} \frac{1}{n^2}$?

Leonhard Euler (1707–1783), in St Petersburg

1731: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to 3 decimal digits.

1733: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to 20 decimal digits.

1734: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to "infinitely many" decimal digits. First proof of the Basel Problem.

1748: Second proof, with more details.

The Basel Problem: a timeline (continued)

1650: P. Mengoli poses the Basel Problem: what is the value of $\sum_{n\geq 1} \frac{1}{n^2}$?

Leonhard Euler (1707–1783), in St Petersburg

1731: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to 3 decimal digits.

1733: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to 20 decimal digits.

1734: computes $\sum_{n\geq 1} \frac{1}{n^2}$ to "infinitely many" decimal digits. First proof of the Basel Problem.

1748: Second proof, with more details.

1997 McKinzie & Tuckey complete Euler's proof, without modern tools, in *Hidden Lemmas in Euler's Summation of the Reciprocals of the Squares*.

Basel Problem: the modern quest for the “simplest proof”

Basel Problem: the modern quest for the "simplest proof"

Trigonometric single integrals or identities: (Yaglom and Yaglom 1953 [101]), (Matsuoka 1961 [64]), (Stark 1969 [92]), (Holme 1970 [45]), (Stark 1970 [93]), (Skau and Selmer 1971 [89]), (Giesy 1972 [36]), (Papadimitriou 1973 [73]), (Stark 1978 [95]), (Stark 1979 [96]), (Ransford 1982 [77]), (Russell 1991 [82]), (Kortram 1996 [56]), (Hofbauer 2002 [44]), (Woodhouse 2007 [100]), (Passare 2008 [74]), (Levrie 2011 [59]), (Benko 2012 [7]), (Daners 2012 [24]), (Muzaffar 2013 [69]), (Brink 2014 [13]), (Glebov 2015c [39]), (Lord 2016 [60]), (Moreno 2016 [66]), (Velleman 2016 [98]), (Pąk and Korniłowicz 2017 [72]), (Siklos 2018 [87]), (Ribeiro 2019 [79]), and (Del Vigna 2023 [25]).

Maclaurin/power/elementary series: (Knopp and Schur 1918 [54]), (Choe 1987 [21]), (Kimble 1987 [52]), (Shea 1988/9 [85]), (Dumont 1992 [26]), (Kalman and McKinzie 2012 [51]), (Benko and Molokach 2013 [8]), (Patyi 2013 [75]), (Krause 2014 [57]), (Vermeeren 2018 [99]), (Silagadze 2019 [88]), (Komornik 2022 [55]), (Campbell 2024a [15]), (Campbell 2024b [16]), (Campbell and Levrie 2024b [18]), (Abreu 2025a [1]), and (Abreu 2025b [2]).

Double integrals: (Goldscheider 1913 [40]), (Apostol 1983 [4]), (Lord 2002 [61]), (Harper 2003 [41]), (Ivan 2008 [47]), (Jameson 2013 [49]), (Jameson and Lord 2013 [50]), (Ritelli 2013 [80]), (Glebov 2015a [37]), (Shiu 2016 [86]), (Novac 2017 [70]), (Pause 2018 [76]), (Murty 2019 [68]), and (Markov 2022 [62]).

Hypergeometric series: (Choi and Rathie 1997 [22]), (Choi, Rathie and Srivastava 1999 [23]), (Campbell 2022 [14]), (Rathie and Lim 2025 [78]), and (Levrie 2026 [58]).

Probability: (Fujita 2008 [35]), (Pace 2011 [71]), (Holst 2013 [46]), and (Aste 2024 [5]).

Complex analysis: (Marshall 2010 [63]), (Glebov 2015b [38]), and (Jain 2024 [48]).

Topology: (Rosenberg 1984 [81]).

Basel Problem: the modern quest for the "simplest proof"

Trigonometric single integrals or identities: (Yaglom and Yaglom 1953 [101]), (Matsuoka 1961 [64]), (Stark 1969 [92]), (Holme 1970 [45]), (Stark 1970 [93]), (Skau and Selmer 1971 [89]), (Giesy 1972 [36]), (Papadimitriou 1973 [73]), (Stark 1978 [95]), (Stark 1979 [96]), (Ransford 1982 [77]), (Russell 1991 [82]), (Kortram 1996 [56]), (Hofbauer 2002 [44]), (Woodhouse 2007 [100]), (Passare 2008 [74]), (Levrie 2011 [59]), (Benko 2012 [7]), (Daners 2012 [24]), (Muzaffar 2013 [69]), (Brink 2014 [13]), (Glebov 2015c [39]), (Lord 2016 [60]), (Moreno 2016 [66]), (Velleman 2016 [98]), (Pąk and Korniłowicz 2017 [72]), (Siklos 2018 [87]), (Ribeiro 2019 [79]), and (Del Vigna 2023 [25]).

Maclaurin/power/elementary series: (Knopp and Schur 1918 [54]), (Choe 1987 [21]), (Kimble 1987 [52]), (Shea 1988/9 [85]), (Dumont 1992 [26]), (Kalman and McKinzie 2012 [51]), (Benko and Molokach 2013 [8]), (Patyi 2013 [75]), (Krause 2014 [57]), (Vermeeren 2018 [99]), (Silagadze 2019 [88]), (Komornik 2022 [55]), (Campbell 2024a [15]), (Campbell 2024b [16]), (Campbell and Levrie 2024b [18]), (Abreu 2025a [1]), and (Abreu 2025b [2]).

Double integrals: (Goldscheider 1913 [40]), (Apostol 1983 [4]), (Lord 2002 [61]), (Harper 2003 [41]), (Ivan 2008 [47]), (Jameson 2013 [49]), (Jameson and Lord 2013 [50]), (Ritelli 2013 [80]), (Glebov 2015a [37]), (Shiu 2016 [86]), (Novac 2017 [70]), (Pause 2018 [76]), (Murty 2019 [68]), and (Markov 2022 [62]).

Hypergeometric series: (Choi and Rathie 1997 [22]), (Choi, Rathie and Srivastava 1999 [23]), (Campbell 2022 [14]), (Rathie and Lim 2025 [78]), and (Levrie 2026 [58]).

Probability: (Fujita 2008 [35]), (Pace 2011 [71]), (Holst 2013 [46]), and (Aste 2024 [5]).

Complex analysis: (Marshall 2010 [63]), (Glebov 2015b [38]), and (Jain 2024 [48]).

Topology: (Rosenberg 1984 [81]).

Mr Judge's favourite proof of the Basel Problem is geometric, by Johan Wästlund in **2010**. 3Blue1Brown made a video: link in the worksheet.

Question 0

Question 0. (Nicole Oresme, ∼1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

This is called *harmonic series*, due to its links to music theory.

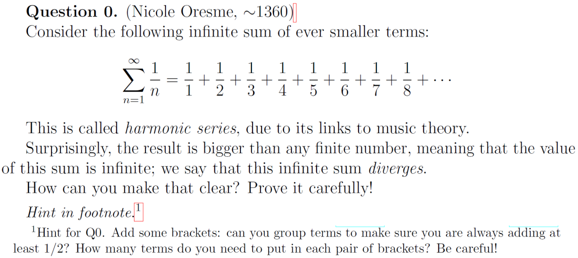
Surprisingly, the result is bigger than any finite number, meaning that the value of this sum is infinite; we say that this infinite sum *diverges*.

How can you make that clear? Prove it carefully!

Hint in footnote.[1]

[1] Hint for Q0. Add some brackets: can you group terms to make sure you are always adding at least 1/2? How many terms do you need to put in each pair of brackets? Be careful!

Question 0



Question 0. (Nicole Oresme, ∼1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

This is called *harmonic series*, due to its links to music theory.

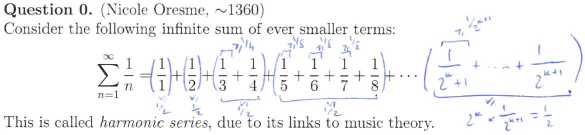
Surprisingly, the result is bigger than any finite number, meaning that the value of this sum is infinite; we say that this infinite sum *diverges*.

How can you make that clear? Prove it carefully!

Hint in footnote.[1]

---

[1] Hint for Q0. Add some brackets: can you group terms to make sure you are always adding at least 1/2? How many terms do you need to put in each pair of brackets? Be careful!

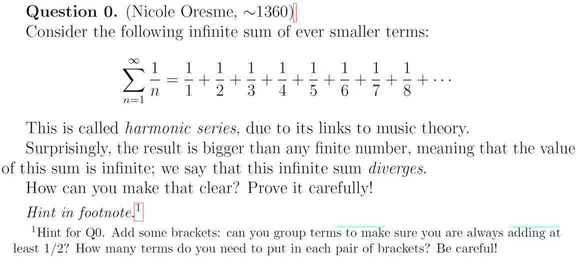


Question 0. (Nicole Oresme, ∼1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n} = \underbrace{\left(\frac{1}{1}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{2}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\overset{\geq \frac{1}{4}}{\frac{1}{3}} + \frac{1}{4}\right)}_{\geq \frac{1}{2}} + \underbrace{\left(\overset{\geq \frac{1}{8}}{\frac{1}{5}} + \overset{\geq \frac{1}{8}}{\frac{1}{6}} + \overset{\geq \frac{1}{8}}{\frac{1}{7}} + \frac{1}{8}\right)}_{\geq \frac{1}{2}} + \cdots \underbrace{\left(\overset{\geq \frac{1}{2^{k+1}}}{\frac{1}{2^k+1}} + \cdots + \frac{1}{2^{k+1}}\right)}_{\geq\, 2^k \times \frac{1}{2^{k+1}} = \frac{1}{2}}$$

This is called *harmonic series*, due to its links to music theory.

Question 0



Question 0. (Nicole Oresme, ~1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

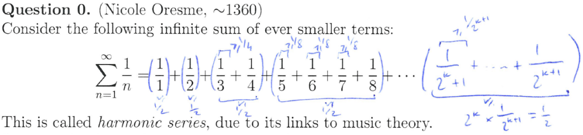
This is called *harmonic series*, due to its links to music theory.

Surprisingly, the result is bigger than any finite number, meaning that the value of this sum is infinite; we say that this infinite sum *diverges*.

How can you make that clear? Prove it carefully!

Hint in footnote.[1]

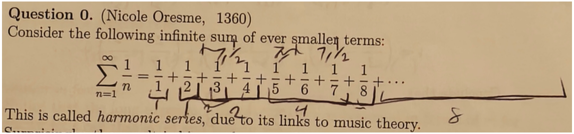
[1] Hint for Q0. Add some brackets: can you group terms to make sure you are always adding at least 1/2? How many terms do you need to put in each pair of brackets? Be careful!



Question 0. (Nicole Oresme, ~1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\underbrace{\left(\frac{1}{1}\right)}_{\geq \frac{1}{2}}+\underbrace{\left(\frac{1}{2}\right)}_{\geq \frac{1}{2}}+\underbrace{\left(\overset{\geq 1/4}{\frac{1}{3}}+\frac{1}{4}\right)}_{\geq \frac{1}{2}}+\underbrace{\left(\overset{\geq 1/8}{\frac{1}{5}}+\overset{\geq 1/8}{\frac{1}{6}}+\overset{\geq 1/8}{\frac{1}{7}}+\frac{1}{8}\right)}_{\geq \frac{1}{2}}+\cdots\underbrace{\left(\overset{\geq 1/2^{k+1}}{\frac{1}{2^k+1}}+\cdots+\frac{1}{2^{k+1}}\right)}_{\geq\, 2^k\times\frac{1}{2^{k+1}}=\frac{1}{2}}$$

This is called *harmonic series*, due to its links to music theory.

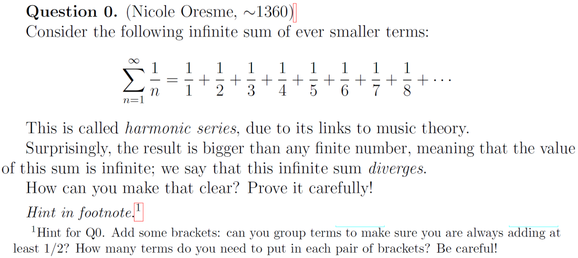


Question 0. (Nicole Oresme, 1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\underbrace{\frac{1}{1}}_{1}+\underbrace{\frac{1}{2}+\overset{\geq 1/2}{\frac{1}{3}}+\frac{1}{4}}_{2}+\underbrace{\overset{\geq 1/2}{\frac{1}{5}}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{4}+\underbrace{\cdots}_{8}$$

This is called *harmonic series*, due to its links to music theory.

Question 0



Question 0. (Nicole Oresme, ∼1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

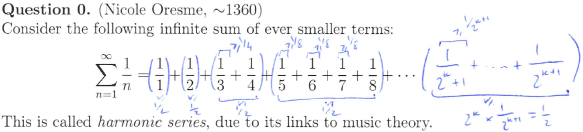
This is called *harmonic series*, due to its links to music theory.

Surprisingly, the result is bigger than any finite number, meaning that the value of this sum is infinite; we say that this infinite sum *diverges*.

How can you make that clear? Prove it carefully!

Hint in footnote.[1]

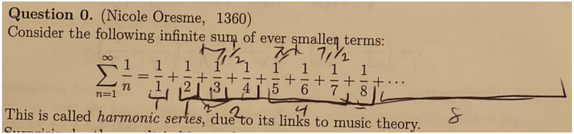
[1] Hint for Q0. Add some brackets: can you group terms to make sure you are always adding at least 1/2? How many terms do you need to put in each pair of brackets? Be careful!



Question 0. (Nicole Oresme, ∼1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\underbrace{\left(\frac{1}{1}\right)}_{\geqslant 1/2}+\underbrace{\left(\frac{1}{2}\right)}_{\geqslant \frac{1}{2}}+\underbrace{\left(\overset{\geqslant 1/4}{\frac{1}{3}}+\frac{1}{4}\right)}_{\geqslant 1/2}+\underbrace{\left(\overset{\geqslant 1/8}{\frac{1}{5}}+\overset{\geqslant 1/8}{\frac{1}{6}}+\overset{\geqslant 1/8}{\frac{1}{7}}+\frac{1}{8}\right)}_{\geqslant 1/2}+\cdots\underbrace{\left(\overset{\geqslant 1/2^{k+1}}{\frac{1}{2^k+1}}+\cdots+\frac{1}{2^{k+1}}\right)}_{\geqslant}$$

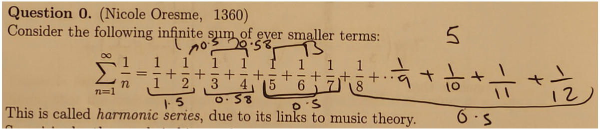
This is called *harmonic series*, due to its links to music theory. $2^k\times\frac{1}{2^{k+1}}=\frac{1}{2}$



Question 0. (Nicole Oresme, 1360)
Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\underbrace{\frac{1}{1}}_{1}+\underbrace{\frac{1}{2}}_{1}+\underbrace{\overset{\geqslant 1/2}{\frac{1}{3}+\frac{1}{4}}}_{2}+\underbrace{\overset{\geqslant}{\frac{1}{5}+\frac{1}{6}}+\overset{\geqslant 1/2}{\frac{1}{7}+\frac{1}{8}}}_{4}+\underbrace{\cdots}_{8}$$

This is called *harmonic series*, due to its links to music theory.



Question 0. (Nicole Oresme, 1360)
Consider the following infinite sum of ever smaller terms: 5

$$\sum_{n=1}^{\infty}\frac{1}{n}=\underbrace{\frac{1}{1}+\frac{1}{2}}_{1\cdot5}+\overset{\geqslant 0\cdot5\quad\geqslant 0\cdot58}{\underbrace{\frac{1}{3}+\frac{1}{4}}_{0\cdot58}}+\overset{\geqslant 3}{\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}_{0\cdot5}}+\underbrace{\frac{1}{8}+\cdots\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}}_{6\cdot5}$$

This is called *harmonic series*, due to its links to music theory.

Observations 0.5 & 1

Observation 0.5. In 1689 Johann Bernoulli proves that $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$. We are so close to the goal, but still so far. Find a modern proof of this as extension at the end of this sheet.

Observation 1. Notice that it is not yet clear whether $\sum_{n\geq 1} \frac{1}{n^2}$ goes off to infinity like $\sum_{n\geq 1} \frac{1}{n}$.

Bernoulli shows that $\sum_{n\geq 1} \frac{1}{n^2} \leq 2$ already in 1689. Goldbach refines that argument in 1728 to obtain $\sum_{n\geq 1} \frac{1}{n^2} \leq 5/3$.

Proof. See extension at the end of this sheet. $\square$

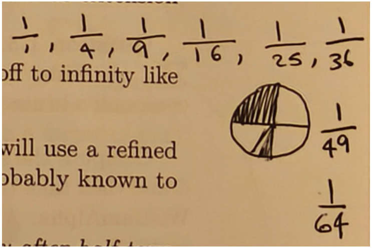
Observations 0.5 & 1

Observation 0.5. In 1689 Johann Bernoulli proves that $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$. We are so close to the goal, but still so far. Find a modern proof of this as extension at the end of this sheet.

Observation 1. Notice that it is not yet clear whether $\sum_{n\geq 1} \frac{1}{n^2}$ goes off to infinity like $\sum_{n\geq 1} \frac{1}{n}$.

Bernoulli shows that $\sum_{n\geq 1} \frac{1}{n^2} \leq 2$ already in 1689. Goldbach refines that argument in 1728 to obtain $\sum_{n\geq 1} \frac{1}{n^2} \leq 5/3$.

Proof. See extension at the end of this sheet. $\square$



$\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}$

ff to infinity like

$\frac{1}{49}$

will use a refined
bably known to

$\frac{1}{64}$

Question 1.5

Question 1.5. Following Observation 1, it now makes sense to ask what's the value of $\sum_{n\geq 1} \frac{1}{n^2}$. Euler took more than 3 years to approximate it ingeniously, but we can use some computer's brute force.

Compute the sum up to the first 5000 terms; you can scan the QR code here to use WolframAlpha.

This is still less accurate than what Euler computed in 1733!

Use this approximation to estimate $\sqrt{6\sum_{n\geq 1} \frac{1}{n^2}} \approx \qquad .$

Thus, make a conjecture: $\sum_{n\geq 1} \frac{1}{n^2} = \qquad .$

Question 1.5

Question 1.5. Following Observation 1, it now makes sense to ask what's the value of $\sum_{n\geq 1}\frac{1}{n^2}$. Euler took more than 3 years to approximate it ingeniously, but we can use some computer's brute force.

Compute the sum up to the first 5000 terms; you can scan the QR code here to use WolframAlpha.

This is still less accurate than what Euler computed in 1733!

Use this approximation to estimate $\sqrt{6\sum_{n\geq 1}\frac{1}{n^2}} \approx \pi$.

Thus, make a conjecture: $\sum_{n\geq 1}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Question 1.5

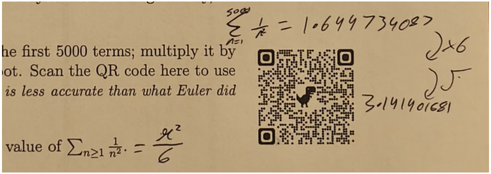
Question 1.5. Following Observation 1, it now makes sense to ask what's the value of $\sum_{n\geq 1}\frac{1}{n^2}$. Euler took more than 3 years to approximate it ingeniously, but we can use some computer's brute force.

Compute the sum up to the first 5000 terms; you can scan the QR code here to use WolframAlpha.

This is still less accurate than what Euler computed in 1733!

Use this approximation to estimate $\sqrt{6\sum_{n\geq 1}\frac{1}{n^2}} \approx \pi$.

Thus, make a conjecture: $\sum_{n\geq 1}\frac{1}{n^2} = \frac{\pi^2}{6}$.



$\sum_{n=1}^{5000}\frac{1}{n^2} = 1.644734087$

he first 5000 terms; multiply it by

ot. Scan the QR code here to use

is less accurate than what Euler did

$\times 6$

$\sqrt{}$

3.141401681

value of $\sum_{n\geq 1}\frac{1}{n^2}. = \frac{\pi^2}{6}$

Fact 2 & Question 2.5

We now need the tools given by the following two results.

Fact 2. (Factor Theorem: 1600s) If p is a polynomial and $p(r) = 0$, then $(x - r)$ is a factor of $p(x)$.

Proof. See extension at the end of this sheet. ☐

Question 2.5. (Useful Consequence) Using Fact 2, prove that if p is a polynomial and for some $r \neq 0$ we have $p(r) = 0$, then $(1 - x/r)$ is a factor of $p(x)$.

Hint in footnote.[2]

---

[2] Hint for Q2.5: What about dividing and multiplying by $-r$?

Fact 2 & Question 2.5

We now need the tools given by the following two results.

Fact 2. (Factor Theorem: 1600s) If p is a polynomial and $p(r) = 0$, then $(x - r)$ is a factor of $p(x)$.

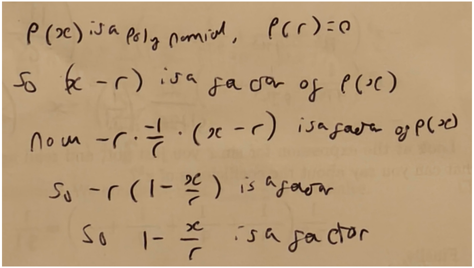
Proof. See extension at the end of this sheet. $\square$

Question 2.5. (Useful Consequence) Using Fact 2, prove that if p is a polynomial and for some $r \neq 0$ we have $p(r) = 0$, then $(1 - x/r)$ is a factor of $p(x)$.

Hint in footnote.[2]

---

[2] Hint for Q2.5: What about dividing and multiplying by $-r$?



$P(x)$ is a Polynomial, $P(r)=0$

So $(x-r)$ is a factor of $P(x)$

now $-r \cdot \frac{-1}{r} \cdot (x-r)$ is a factor of $P(x)$

So $-r\left(1-\frac{x}{r}\right)$ is a factor

So $1-\frac{x}{r}$ is a factor

Fact 2 & Question 2.5

We now need the tools given by the following two results.

Fact 2. (Factor Theorem: 1600s) If p is a polynomial and $p(r) = 0$, then $(x - r)$ is a factor of $p(x)$.

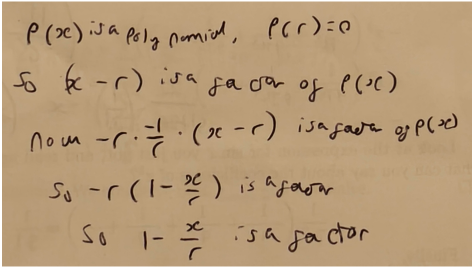
Proof. See extension at the end of this sheet. ☐

Question 2.5. (Useful Consequence) Using Fact 2, prove that if p is a polynomial and for some $r \neq 0$ we have $p(r) = 0$, then $(1 - x/r)$ is a factor of $p(x)$.

Hint in footnote.[2]

---

[2] Hint for Q2.5: What about dividing and multiplying by $-r$?



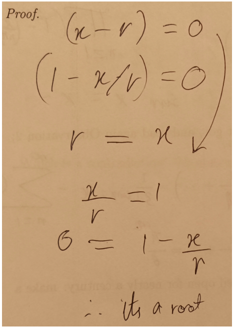
$P(x)$ is a Polynomial, $P(r)=0$

So $(x-r)$ is a factor of $P(x)$

Now $-r \cdot \frac{-1}{r} \cdot (x-r)$ is a factor of $P(x)$

So $-r(1-\frac{x}{r})$ is a factor

So $1-\frac{x}{r}$ is a factor



Proof.

$$(x-r)=0$$

$$(1-x/r)=0$$

$$r=x$$

$$\frac{x}{r}=1$$

$$0=1-\frac{x}{r}$$

$\therefore$ its a root

Observation 3 & Question 4

Observation 3. Euler knew the formula for sine found by Isaac Newton in the 1660s:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

He took this formula as a starting point.

Question 4. List the *seven* roots of $\sin x$ between -3π and 3π in the left column of the table on the right.

Now, let's treat $\sin x$ like a polynomial.

For each root you put in the table, write in the second column what is the corresponding "factor" of $\sin x$ given by the Useful Consequence (Question 2.5) above.

For *one* root you cannot use the Useful Consequence to obtain the corresponding factor: use the Factor Theorem (Fact 2) instead.

roots of $\sin x$	factors of $\sin x$

Observation 3 & Question 4

Observation 3. Euler knew the formula for sine found by Isaac Newton in the 1660s:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

He took this formula as a starting point.

Question 4. List the *seven* roots of $\sin x$ between -3π and 3π in the left column of the table on the right.

Now, let's treat $\sin x$ like a polynomial.

For each root you put in the table, write in the second column what is the corresponding "factor" of $\sin x$ given by the Useful Consequence (Question 2.5) above.

For *one* root you cannot use the Useful Consequence to obtain the corresponding factor: use the Factor Theorem (Fact 2) instead.

roots of $\sin x$	factors of $\sin x$
-3π	$\left(1+\frac{x}{3\pi}\right)$
-2π	$\left(1+\frac{x}{2\pi}\right)$
$-\pi$	$\left(1+\frac{x}{\pi}\right)$
0	x
π	$\left(1-\frac{x}{\pi}\right)$
2π	$\left(1-\frac{x}{2\pi}\right)$
3π	$\left(1-\frac{x}{3\pi}\right)$

Observation 3 & Question 4

Observation 3. Euler knew the formula for sine found by Isaac Newton in the 1660s:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

He took this formula as a starting point.

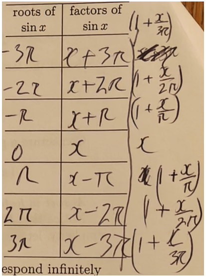
Question 4. List the *seven* roots of $\sin x$ between -3π and 3π in the left column of the table on the right.

Now, let's treat $\sin x$ like a polynomial.

For each root you put in the table, write in the second column what is the corresponding "factor" of $\sin x$ given by the Useful Consequence (Question 2.5) above.

For *one* root you cannot use the Useful Consequence to obtain the corresponding factor: use the Factor Theorem (Fact 2) instead.

roots of $\sin x$	factors of $\sin x$
-3π	$\left(1+\frac{x}{3\pi}\right)$
-2π	$\left(1+\frac{x}{2\pi}\right)$
$-\pi$	$\left(1+\frac{x}{\pi}\right)$
0	x
π	$\left(1-\frac{x}{\pi}\right)$
2π	$\left(1-\frac{x}{2\pi}\right)$
3π	$\left(1-\frac{x}{3\pi}\right)$



roots of $\sin x$	factors of $\sin x$	$\left(1+\frac{x}{3\pi}\right)$
-3π	$x+3\pi$	
-2π	$x+2\pi$	$\left(1+\frac{x}{2\pi}\right)$
$-\pi$	$x+\pi$	$\left(1+\frac{x}{\pi}\right)$
0	x	x
π	$x-\pi$	$\left(1+\frac{x}{\pi}\right)$
2π	$x-2\pi$	$\left(1+\frac{x}{2\pi}\right)$
3π	$x-3\pi$	$\left(1+\frac{x}{3\pi}\right)$

espond infinitely

Observation 3 & Question 4

Observation 3. Euler knew the formula for sine found by Isaac Newton in the 1660s:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

He took this formula as a starting point.

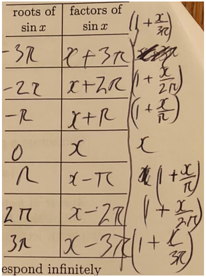
Question 4. List the *seven* roots of $\sin x$ between -3π and 3π in the left column of the table on the right.

Now, let's treat $\sin x$ like a polynomial.

For each root you put in the table, write in the second column what is the corresponding "factor" of $\sin x$ given by the Useful Consequence (Question 2.5) above.

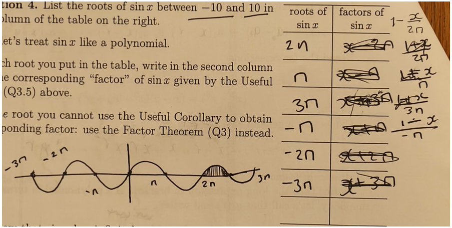
For *one* root you cannot use the Useful Consequence to obtain the corresponding factor: use the Factor Theorem (Fact 2) instead.

roots of $\sin x$	factors of $\sin x$
-3π	$\left(1+\frac{x}{3\pi}\right)$
-2π	$\left(1+\frac{x}{2\pi}\right)$
$-\pi$	$\left(1+\frac{x}{\pi}\right)$
0	x
π	$\left(1-\frac{x}{\pi}\right)$
2π	$\left(1-\frac{x}{2\pi}\right)$
3π	$\left(1-\frac{x}{3\pi}\right)$



roots of $\sin x$	factors of $\sin x$	$\left(1+\frac{x}{3\pi}\right)$
-3π	$x+3\pi$	~~$x+3\pi$~~
-2π	$x+2\pi$	$\left(1+\frac{x}{2\pi}\right)$
$-\pi$	$x+\pi$	$\left(1+\frac{x}{\pi}\right)$
0	x	x
π	$x-\pi$	~~x~~ $\left(1+\frac{x}{\pi}\right)$
2π	$x-2\pi$	$\left(1+\frac{x}{2\pi}\right)$
3π	$x-3\pi$	$\left(1+\frac{x}{3\pi}\right)$

espond infinitely



tion 4. List the roots of $\sin x$ between -10 and 10 in
olumn of the table on the right.

et's treat $\sin x$ like a polynomial.

ch root you put in the table, write in the second column
e corresponding "factor" of $\sin x$ given by the Useful
(Q3.5) above.

e root you cannot use the Useful Corollary to obtain
ponding factor: use the Factor Theorem (Q3) instead.

roots of $\sin x$	factors of $\sin x$	$1-\frac{x}{2\pi}$
2π	~~$x-2\pi$~~	$\frac{~~1-x~~}{2\pi}$
π	~~$x-\pi$~~	$\frac{~~1-~~x}{\pi}$
3π	~~$x-3\pi$~~	$\frac{~~1-~~x}{3\pi}$
$-\pi$	~~$x+\pi$~~	$\frac{1-x}{-\pi}$
-2π	~~$x+2\pi$~~	
-3π	~~$x+3\pi$~~	

Question 4

You know that $\sin x$ has infinitely many roots, to which correspond infinitely many factors of $\sin x$.

Write $\sin x$ as the product of its infinitely many factors:

$$\sin x = (\quad)\left(1 - \quad\right)\left(1 + \quad\right)\left(1 - \quad\right)\left(1 + \quad\right)\left(1 - \quad\right)\left(1 + \quad\right)\cdots$$

$$= (\quad)\left(1^2 - \quad\right)\left(1^2 - \quad\right)\left(1^2 - \quad\right)\cdots$$

Expand the product above and rearrange the terms so to express a "polynomial" of infinite degree.

Hint in footnote.[3]

$$\begin{aligned}\sin x = {} & x \\ & - x^3\left(\frac{1}{\quad\quad} + \frac{1}{\quad\quad} + \frac{1}{\quad\quad} + \cdots\right) \\ & + x^5\left(\frac{1}{\quad\quad} + \frac{1}{\quad\quad} + \cdots\right) \\ & - \cdots\end{aligned}$$

Look at the expression for $\sin x$ you just got, and read again Observation 3; what can you say about the coefficients of x^3?

$$\frac{1}{\quad^{2}}\left(\frac{1}{\quad\quad} + \frac{1}{\quad\quad} + \frac{1}{\quad\quad} + \cdots\right) = \frac{1}{\quad !}$$

Finally,

$$1 + \frac{1}{\quad\quad} + \frac{1}{\quad\quad} + \frac{1}{\quad\quad} + \cdots = \underline{\quad\quad}$$

You answered a question that had remained open for nearly a century: make a celebratory dance!

---

[3] Hint for Q4. Consider the product of the first *two* terms $(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

Then, consider the product of the *three* terms $(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

Finally, consider $(\quad)(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: what are the coefficients of the x and x^3 terms?

Question 4

You know that $\sin x$ has infinitely many roots, to which correspond infinitely many factors of $\sin x$.

Write $\sin x$ as the product of its infinitely many factors:

$$\sin x = (x)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right) \cdots$$

$$= (x)\left(1^2 - \frac{x^2}{\pi^2}\right)\left(1^2 - \frac{x^2}{4\pi^2}\right)\left(1^2 - \frac{x^2}{9\pi^2}\right) \cdots$$

Expand the product above and rearrange the terms so to express a "polynomial" of infinite degree.

Hint in footnote.[3]

$$\begin{aligned} \sin x = {} & x \\ & - x^3\left(\frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} + \cdots\right) \\ & + x^5\left(\frac{1}{\quad} + \frac{1}{\quad} + \cdots\right) \\ & - \cdots \end{aligned}$$

Look at the expression for $\sin x$ you just got, and read again Observation 3; what can you say about the coefficients of x^3?

$$\frac{1}{\quad^2}\left(\frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} + \cdots\right) = \frac{1}{\quad !}$$

Finally,

$$1 + \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad} + \cdots = \frac{\quad}{\quad}$$

You answered a question that had remained open for nearly a century: make a celebratory dance!

---

[3] Hint for Q4. Consider the product of the first *two* terms $(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

Then, consider the product of the *three* terms $(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

Finally, consider $(\quad)(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: what are the coefficients of the x and x^3 terms?

Question 4

You know that $\sin x$ has infinitely many roots, to which correspond infinitely many factors of $\sin x$.

Write $\sin x$ as the product of its infinitely many factors:

$$\sin x = (x)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right) \cdots$$

$$= (x)\left(1^2 - \frac{x^2}{\pi^2}\right)\left(1^2 - \frac{x^2}{4\pi^2}\right)\left(1^2 - \frac{x^2}{9\pi^2}\right) \cdots$$

Expand the product above and rearrange the terms so to express a "polynomial" of infinite degree.

Hint in footnote.[3]

$$\begin{aligned}\sin x = \; & x \\ & - x^3\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right) \\ & + x^5\left(\frac{1}{4\pi^4} + \frac{1}{9\pi^4} + \cdots\right) \\ & - \;\cdots\end{aligned}$$

Look at the expression for $\sin x$ you just got, and read again Observation 3; what can you say about the coefficients of x^3?

$$\frac{1}{^{2}}\left(\frac{1}{} + \frac{1}{} + \frac{1}{} + \cdots\right) = \frac{1}{!}$$

Finally,

$$1 + \frac{1}{} + \frac{1}{} + \frac{1}{} + \cdots = \underline{}$$

You answered a question that had remained open for nearly a century: make a celebratory dance!

---

[3] Hint for Q4. Consider the product of the first *two* terms $(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?
Then, consider the product of the *three* terms $(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?
Finally, consider $(\quad)(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: what are the coefficients of the x and x^3 terms?

Question 4

You know that $\sin x$ has infinitely many roots, to which correspond infinitely many factors of $\sin x$.

Write $\sin x$ as the product of its infinitely many factors:

$$\sin x = (x)\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots$$

$$= (x)\left(1^2-\frac{x^2}{\pi^2}\right)\left(1^2-\frac{x^2}{4\pi^2}\right)\left(1^2-\frac{x^2}{9\pi^2}\right)\cdots$$

Expand the product above and rearrange the terms so to express a "polynomial" of infinite degree.

Hint in footnote.[3]

$$\begin{aligned}\sin x = {} & x \\ & - x^3\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right) \\ & + x^5\left(\frac{1}{4\pi^4}+\frac{1}{9\pi^4}+\cdots\right) \\ & - \cdots\end{aligned}$$

Look at the expression for $\sin x$ you just got, and read again Observation 3; what can you say about the coefficients of x^3?

$$\frac{1}{\pi^2}\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots\right)=\frac{1}{3!}$$

Finally,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}$$

You answered a question that had remained open for nearly a century: make a celebratory dance!

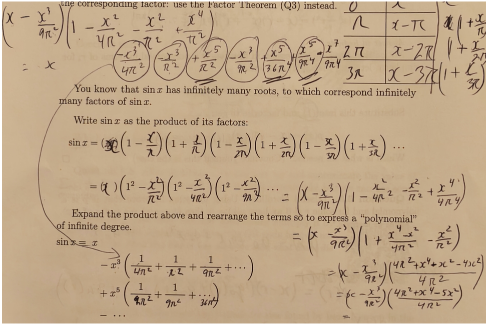
---

[3] Hint for Q4. Consider the product of the first *two* terms $(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

Then, consider the product of the *three* terms $(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

Finally, consider $(\quad)(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: what are the coefficients of the x and x^3 terms?

Question 4



the corresponding factor: use the Factor Theorem (Q3) instead.

$$\left(x-\frac{x^3}{9\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}-\frac{x^2}{\pi^2}+\frac{x^4}{\pi^2}\right)$$

$$= x \quad -\frac{x^3}{4\pi^2} \quad -\frac{x^3}{\pi^2} \quad +\frac{x^5}{\pi^2} \quad -\frac{x^3}{9\pi^2} \quad +\frac{x^5}{36\pi^4} + \frac{x^5}{9\pi^4} - \frac{x^7}{9\pi^4}$$

0	x
π	$x-\pi$
2π	$x-2\pi$
3π	$x-3\pi$

$x\left(1+\frac{x}{\pi}\right)$

$\left(1+\frac{x}{2\pi}\right)$

$\left(1+\frac{x}{3\pi}\right)$

You know that $\sin x$ has infinitely many roots, to which correspond infinitely many factors of $\sin x$.

Write $\sin x$ as the product of its factors:

$$\sin x = (x)\left(1-\frac{x}{\pi}\right)\left(1+\frac{1}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots$$

$$= (x)\left(1^2-\frac{x^2}{\pi^2}\right)\left(1^2-\frac{x^2}{4\pi^2}\right)\left(1^2-\frac{x^2}{9\pi^2}\right)\cdots = \left(x-\frac{x^3}{9\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}-\frac{x^2}{\pi^2}+\frac{x^4}{4\pi^4}\right)$$

Expand the product above and rearrange the terms so to express a "polynomial" of infinite degree.

$$= \left(x-\frac{x^3}{9\pi^2}\right)\left(1+\frac{x^4-x^2}{4\pi^2}-\frac{x^2}{\pi^2}\right)$$

$$= \left(x-\frac{x^3}{9\pi^2}\right)\left(\frac{4\pi^2+x^4+x^2-4x^2}{4\pi^2}\right)$$

$$= \left(x-\frac{x^3}{9\pi^2}\right)\left(\frac{4\pi^2+x^4-5x^2}{4\pi^2}\right)$$

$=$

$$\sin x = x$$
$$-x^3\left(\frac{1}{4\pi^2}+\frac{1}{\pi^2}+\frac{1}{9\pi^2}+\cdots\right)$$
$$+x^5\left(\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{\cdots}{36\pi^4}\right)$$
$$-\cdots$$

You know that $\sin x$ has infinitely many roots, to which correspond infinitely many factors of $\sin x$.

Write $\sin x$ as the product of its infinitely many factors:

$$\sin x = (x)\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots$$

$$= (x)\left(1^2-\frac{x^2}{\pi^2}\right)\left(1^2-\frac{x^2}{4\pi^2}\right)\left(1^2-\frac{x^2}{9\pi^2}\right)\cdots$$

Expand the product above and rearrange the terms so to express a "polynomial" of infinite degree.

Hint in footnote.[3]

$$\sin x = x$$
$$-x^3\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right)$$
$$+x^5\left(\frac{1}{4\pi^4}+\frac{1}{9\pi^4}+\cdots\right)$$
$$-\cdots$$

Look at the expression for $\sin x$ you just got, and read again Observation 3; what can you say about the coefficients of x^3?

$$\frac{1}{\pi^2}\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots\right)=\frac{1}{3!}$$

Finally,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}$$

You answered a question that had remained open for nearly a century: make a celebratory dance!

---

[3] Hint for Q4. Consider the product of the first *two* terms $(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

Then, consider the product of the *three* terms $(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: in the result, what are the coefficients of the constant and x^2 terms?

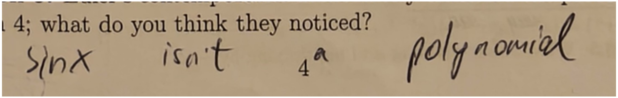
Finally, consider $(\quad)(1^2 - \quad)(1^2 - \quad)(1^2 - \quad)$: what are the coefficients of the x and x^3 terms?

Question 5

Question 5. Euler's contemporaries immediately criticized *two* steps in the proof you wrote in Question 4; what do you think they noticed?

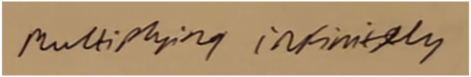
Question 5

Question 5. Euler's contemporaries immediately criticized *two* steps in the proof you wrote in Question 4; what do you think they noticed?



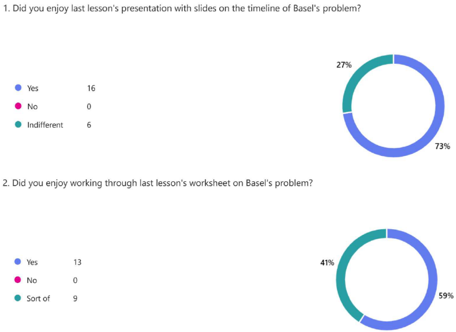
4; what do you think they noticed?

sinx isn't a polynomial



multiplying infinitely

Anonymous students' feedback



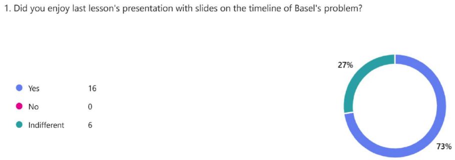
1. Did you enjoy last lesson's presentation with slides on the timeline of Basel's problem?

● Yes	16
● No	0
● Indifferent	6

2. Did you enjoy working through last lesson's worksheet on Basel's problem?

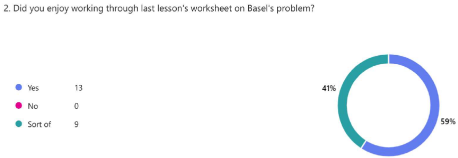
● Yes	13
● No	0
● Sort of	9

Anonymous students' feedback



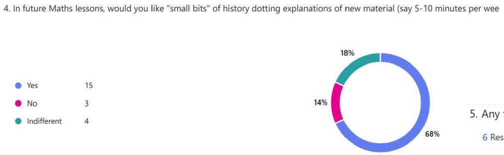
1. Did you enjoy last lesson's presentation with slides on the timeline of Basel's problem?

●	Yes	16
●	No	0
●	Indifferent	6



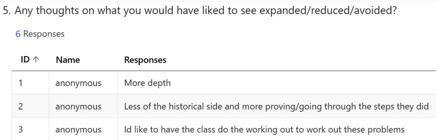
2. Did you enjoy working through last lesson's worksheet on Basel's problem?

●	Yes	13
●	No	0
●	Sort of	9



4. In future Maths lessons, would you like "small bits" of history dotting explanations of new material (say 5-10 minutes per wee

●	Yes	15
●	No	3
●	Indifferent	4



5. Any thoughts on what you would have liked to see expanded/reduced/avoided?

6 Responses

ID ↑	Name	Responses
1	anonymous	More depth
2	anonymous	Less of the historical side and more proving/going through the steps they did
3	anonymous	Id like to have the class do the working out to work out these problems

Possible extensions

Generalisation to Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Possible extensions

Generalisation to Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Linking on to the Riemann Hypothesis, which gained importance as one of Hilbert's 23 Problems, and still remains an unsolved Millennium Problem.

A consequence of the Riemann Hypothesis: the Prime Number Theorem, describing the frequency of primes in the number line.

Possible extensions

Generalisation to Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Linking on to the Riemann Hypothesis, which gained importance as one of Hilbert's 23 Problems, and still remains an unsolved Millennium Problem.

A consequence of the Riemann Hypothesis: the Prime Number Theorem, describing the frequency of primes in the number line.

... and continuing this project with more historically-informed resources!

Thank you!

Davide Leonessi
PGCE in Maths at King's College London
(Teacher of Maths at 1729 Maths School)

`davide.leonessi@kcl.ac.uk`

Resources available at:
`leonessi.org`

Worksheet Sources

- 3Blue1Brown. (2018, March 2). *Why is pi here? And why is it squared? A geometric answer to the Basel problem.* YouTube. `https://www.youtube.com/watch?v=d-o3eB9sfls`
- Ayoub, R. (1974). Euler and the Zeta Function. *The American Mathematical Monthly, 81*(10), 1067–1086. `https://doi.org/10.1080/00029890.1974.11993738`
- Campbell, J., & Levrie, P. (2026). The Modern History of the Basel Problem. *Euleriana, 6*(1), 70–96. `https://doi.org/10.56031/2693-9908.1106`
- Dunham, W. (1987). The Bernoullis and the Harmonic Series. *The College Mathematics Journal, 18*(1), 18–23. `https://doi.org/10.1080/07468342.1987.11973001`
- McKinzie, M., & Tuckey, C. (1997). Hidden lemmas in Euler's summation of the reciprocals of the squares. *Archive for History of Exact Sciences, 51*(1), 29–57. `https://doi.org/10.1007/BF00376450`
- Wästlund, J. (2010). *Summing inverse squares by euclidean geometry.* `https://www.math.chalmers.se/~wastlund/Cosmic.pdf`

Bibliography

- Blåsjö, V. (2014). A Critique of the Modern Consensus in the Historiography of Mathematics. *Journal of Humanistic Mathematics, 4*(2), 113–123. `https://doi.org/10.5642/jhummath.201402.12`
- Freudenthal, H. (2002). Revisiting Mathematics Education: China Lectures. In *Revisiting Mathematics Education*. Kluwer Academic Publishers. `https://doi.org/10.1007/0-306-47202-3`
- Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261. `https://doi.org/10.1007/S10649-008-9174-9`
- Semadeni, Z. (2015). Educational aspects of cognitive transgressions in mathematics. In A. K. Żeromska (Ed.), *Mathematical transgressions and education* (pp. 25–42). Wydawnictwo Szkolne Omega.
- Severi, F. (1930). *Elementi di geometria*. Vallecchi.
- Watford, M., & Clark, K. M. (2024). Investigating transgressive actions undertaken by students while studying mathematics via primary sources. *International Electronic Journal of Mathematics Education, 19*(4). `https://doi.org/10.29333/iejme/15477`