

The Basel Problem:

The quest to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Question 0. (Nicole Oresme, ~1360)

Consider the following infinite sum of ever smaller terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

This is called *harmonic series*, due to its links to music theory.

Surprisingly, the result is bigger than any finite number, meaning that the value of this sum is infinite; we say that this infinite sum *diverges*.

How can you make that clear? Prove it carefully!

*Hint in footnote.*¹

Observation 0.5. In 1689 Johann Bernoulli proves that $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$. We are so close to the goal, but still so far. Find a modern proof of this as extension at the end of this sheet.

Observation 1. Notice that it is not yet clear whether $\sum_{n \geq 1} \frac{1}{n^2}$ goes off to infinity like $\sum_{n \geq 1} \frac{1}{n}$.

Bernoulli shows that $\sum_{n \geq 1} \frac{1}{n^2} \leq 2$ already in 1689. Goldbach refines that argument in 1728 to obtain $\sum_{n \geq 1} \frac{1}{n^2} \leq 5/3$.

Proof. See extension at the end of this sheet. □

Question 1.5. Following Observation 1, it now makes sense to ask what's the value of $\sum_{n \geq 1} \frac{1}{n^2}$. Euler took more than 3 years to approximate it ingeniously, but we can use some computer's brute force.

Compute the sum up to the first 5000 terms; you can scan the QR code here to use WolframAlpha.

This is still less accurate than what Euler computed in 1733!

Use this approximation to estimate $\sqrt{6 \sum_{n \geq 1} \frac{1}{n^2}} \approx$.



Thus, make a conjecture: $\sum_{n \geq 1} \frac{1}{n^2} =$.

This worksheet was made by Davide Leonessi while completing his PGCE at King's College London. An earlier version was used for a Y12 lesson at KCLMS; more information at this link.

¹Hint for Q0. Add some brackets: can you group terms to make sure you are always adding at least 1/2? How many terms do you need to put in each pair of brackets? Be careful!

We now need the tools given by the following two results.

Fact 2. (Factor Theorem: 1600s) If p is a polynomial and $p(r) = 0$, then $(x - r)$ is a factor of $p(x)$.

Proof. See extension at the end of this sheet. □

Question 2.5. (Useful Consequence) Using Fact 2, prove that if p is a polynomial and for some $r \neq 0$ we have $p(r) = 0$, then $(1 - x/r)$ is a factor of $p(x)$.

*Hint in footnote.*²

Proof.

□

Observation 3. Euler knew the formula for sine found by Isaac Newton in the 1660s:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

He took this formula as a starting point.

Question 4. List the *seven* roots of $\sin x$ between -3π and 3π in the left column of the table on the right.

Now, let's treat $\sin x$ like a polynomial.

For each root you put in the table, write in the second column what is the corresponding "factor" of $\sin x$ given by the Useful Consequence (Question 2.5) above.

For *one* root you cannot use the Useful Consequence to obtain the corresponding "factor:" use the Factor Theorem (Fact 2) instead.

roots of $\sin x$	"factors" of $\sin x$

²Hint for Q2.5: What about dividing and multiplying by $-r$?

You know that $\sin x$ has infinitely many roots, to which correspond infinitely many “factors” of $\sin x$.

Write $\sin x$ as the product of its infinitely many “factors:”

$$\begin{aligned} \sin x &= () \left(1 - \right) \left(1 + \right) \left(1 - \right) \left(1 + \right) \left(1 - \right) \left(1 + \right) \cdots \\ &= () \left(1^2 - \right) \left(1^2 - \right) \left(1^2 - \right) \cdots \end{aligned}$$

Expand the product above and rearrange the terms so to express a “polynomial” of infinite degree.

*Hint in footnote.*³

$$\begin{aligned} \sin x &= x \\ &\quad - x^3 \left(\frac{1}{2} + \frac{1}{24} + \frac{1}{720} + \cdots \right) \\ &\quad + x^5 \left(\frac{1}{24} + \frac{1}{720} + \cdots \right) \\ &\quad - \cdots \end{aligned}$$

Look at the expression for $\sin x$ you just got, and read again Observation 3; what can you say about the coefficients of x^3 ?

$$\frac{1}{2} \left(\frac{1}{2} + \frac{1}{24} + \frac{1}{720} + \cdots \right) = \frac{1}{6}$$

Finally,

$$1 + \frac{1}{24} + \frac{1}{720} + \frac{1}{5040} + \cdots = \frac{5}{24}$$

You answered a question that had remained open for nearly a century: make a celebratory dance!

Question 5. Euler’s contemporaries immediately criticized *two* steps in the proof you wrote in Question 4; what do you think they noticed?

³Hint for Q4. Consider the product of the first *two* terms $(1^2 - \frac{1}{2})(1^2 - \frac{1}{24})$: in the result, what are the coefficients of the constant and x^2 terms?

Then, consider the product of the *three* terms $(1^2 - \frac{1}{2})(1^2 - \frac{1}{24})(1^2 - \frac{1}{720})$: in the result, what are the coefficients of the constant and x^2 terms?

Finally, consider $(x)(1^2 - \frac{1}{2})(1^2 - \frac{1}{24})(1^2 - \frac{1}{720})$: what are the coefficients of the x and x^3 terms?

Extension to Question 0. *Alternative proof by contradiction from 1647, by Pietro Mengoli.*

Proof. Suppose for a contradiction that $\sum_{n \geq 1} \frac{1}{n} = S$, for some finite $S > 0$.

Observe that:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} &= \frac{1}{1} + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \dots \\ &= 1 + \sum_{k=1}^{\infty} \left(\frac{1}{-} + \frac{1}{+} + \frac{1}{+} \right) \end{aligned}$$

Now note that, for $m > 1$,

$$\begin{aligned} \frac{1}{m-1} + \frac{1}{m+1} &= \frac{1}{m-1} + \frac{1}{m+1} \\ &= \frac{1}{-\frac{1}{m}} \\ &\geq \frac{2}{m} \quad (\text{Justify : }) \end{aligned}$$

Thus, for $m > 1$,

$$\frac{1}{m-1} + \frac{1}{m} + \frac{1}{m+1} \geq \frac{1}{m}.$$

Hence, from above:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{-} + \frac{1}{+} + \frac{1}{+} \right) \\ &\geq 1 + \sum_{k=1}^{\infty} \frac{3}{m} \\ &= 1 + \end{aligned}$$

Which is a contradiction! We conclude that $\sum_{n \geq 1} \frac{1}{n}$ is not finite. □

Extension to Observation 0.5. Show that $\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1$.

Proof. First, we look at the partial sums; prove by induction on n that

$$\sum_{k=1}^n \frac{1}{k^2+k} = \frac{n}{n+1}.$$

Then, taking the limit for n that goes to infinity, we get:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k^2+k} = \lim_{n \rightarrow \infty} \frac{n}{n+1}$$

And so we conclude:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1$$

□

Observe that this is a modern proof because, back in 1689, Jacob Bernoulli did not have at his disposal formal proofs by induction, nor limits to infinity, nor the handy summation notation. His proof, in addition to being written in Latin, is more involved, with informal “juggling” of infinite sums.

Extension to Observation 1. Show that $\sum_{n \geq 1} \frac{1}{n^2} \leq \frac{5}{3}$

The following argument was probably known to Goldbach, and thus to Euler, in 1728.⁴ Note that this proof uses telescopic sums.

First, notice that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

⁴Goldbach mentions in a 1728 letter to Daniel Bernoulli that he had found the bounds $1 + 16/25 < \sum_{n \geq 1} \frac{1}{n^2} < 1 + 2/3$, without providing an explicit argument; Daniel Bernoulli and Euler both lived in St Petersburg at the time.

Observe that

$$n^2 \geq n^2 - \frac{1}{4} = \left(n - \frac{1}{2}\right) \left(n + \frac{1}{2}\right).$$

So,

$$\begin{aligned} \frac{1}{n^2} &\leq \frac{1}{n^2 - \frac{1}{4}} \\ &= \frac{1}{\left(n - \frac{1}{2}\right) \left(n + \frac{1}{2}\right)} \\ &= \frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}} \end{aligned}$$

Summing from $n = 2$ to infinity:

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n^2} &\leq \sum_{n=2}^{\infty} \left(\frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \end{aligned}$$

Conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \frac{1}{2}$$

Finally,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq \frac{3}{2}$$

Which is the bound we wanted.

Extension to Fact 2. (Factor Theorem: 1600s) Prove that if p is a polynomial and $p(r) = 0$, then $(x - r)$ is a factor of $p(x)$.

Proof. Let

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

So,

$$p(y) =$$

Thus,

$$p(x) - p(y) =$$

Letting $y = r$ for some r such that $p(r) = 0$, deduce that

$$p(x) - p(r) = p(x) = a_1(x - r) + a_2(x - r)^2 + a_3(x - r)^3 + \dots + a_n(x - r)^n \quad (1)$$

Recall the factorisations:

$$\begin{aligned}(x^2 - r^2) &= (x - r)(x + r) \\ (x^3 - r^3) &= (x - r)(x^2 + xr + r^2)\end{aligned}$$

Conjecture a general rule:

$$(x^k - r^k) = (x - r)(x^{k-1} + x^{k-2}r + x^{k-3}r^2 + \dots + r^{k-1}) \quad (2)$$

Answer in footnote.⁵

Notice that, for each k , the long expression in brackets is a polynomial in terms of r ; for simplicity, let's call that $q_k(r)$ and rewrite (2) as:

$$(x^k - r^k) = (x - r)q_k(r)$$

Substitute this into (1) to get:

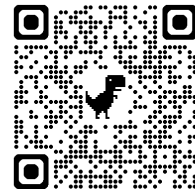
$$p(x) = a_1(x - r) + a_2(x - r)q_2(r) + a_3(x - r)q_3(r) + \dots + a_n(x - r)q_n(r)$$

Finally, factorise:

$$p(x) = (x - r)(a_1 + a_2q_2(r) + a_3q_3(r) + \dots + a_nq_n(r))$$

Which is what we needed: convince yourself this is the case. □

Extension 5. Watch the 17-minute video of 3Blue1Brown's wonderful explanation of the purely geometric solution to the Basel Problem recently given by Johan Wästlund in *Summing inverse squares by euclidean geometry* (2010); access it through the QR code here.



⁵It can be proved by induction that $(x^k - r^k) = (x - r)(x^{k-1} + x^{k-2}r + x^{k-3}r^2 + \dots + r^{k-1})$.